The Mathematics of Savings and Retirement Planning

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Abstract

Saving and retirement planning currently rely on techniques that have proven to be failure prone at best, and completely inadequate under the worst circumstances. Further, existing methods do not provide an understanding of the drivers behind the success or failure of savings plans. This lack of insight, combined with a deficient framework, contributes to the adverse outcomes experienced by many pension plans and individuals.

In this paper, we will introduce a new perspective on modeling savings, that borrows extensively from the field of hydrology. To accomplish this, we will apply a modified version of mathematical techniques used by hydrologists in dam building, to savings and retirement planning. Our goal is to present an intuitive decision framework, based on a novel concept. In the process, we hope to open a new path in the dialogue to improve the state of the art in savings and retirement planning.
“Apathy is one of the characteristic responses of any living organization when it is subject to stimuli too intense or too complicated to cope with.”

- John Dos Pasos

**Introduction**

Building wealth for life’s major events is accompanied by significant uncertainty. Historically, individual and institutional efforts to tame this uncertainty have produced largely disappointing results. We speculate that is one of the primary reasons why defined benefit pension plans are eager to transfer saving responsibility to individuals and their advisors, or to the government.¹

Individuals have not fared any better in this arena. Consider these dire statistics. It is estimated that only one third of U.S. households are saving at levels close to what they will need to retire and sustain an adequate standard of living. It is further estimated that 50% of households are not saving at all for their retirement.²

With failing pension plans and woefully inadequate individual savings, is there any reason for optimism? Perhaps. Saving for retirement, or for that matter, saving for any significant goal, is not an intractable problem. Individuals can achieve their savings goals to the extent that they have proper guidance from the financial industry. That guidance is contingent upon the industry learning from the mistakes of corporate sponsored pension plans. It’s time for the experts to abandon overly simplistic assumptions and failure prone methods, and explore better techniques. Only then will we
further the ability to estimate retirement wealth requirements, and push forward the state of the retirement planning science.

In this paper we will develop closed form mathematical formulas, albeit with many simplifying assumptions, that will shed light on the critical levers of savings and wealth building. The objective of this mathematical framework is to provide insight that has been missing from this challenging task. In the process, we hope to raise the level of knowledge and reduce the apathy factor.

**Using nature as a guide for retirement planning**

In 1906, Harold Edwin Hurst, a British civil servant, was sent to Cairo, Egypt to tame the millennia old problems of the Nile river: floods and drought. The Nile was unpredictable and, although the delta was the most fertile in the world, it was also susceptible to tremendous variations in water levels. The result was either abundance or famine. Hurst’s mission was to control this variation by building a series of dams that would even out the economic impact of the Nile floods. It took forty years for Hurst to solve the puzzle of the Nile. His findings were published in his seminal paper (Hurst 1951).

Hurst’s critical insight was that it is not just the severity of floods that matters; it is also the precise sequence of multiple floods (Mandelbrot 2004). Up to that point, it was assumed that variations in water levels were independent and mimicked a random walk. Hurst demonstrated that the variations in the Nile and other rivers displayed long term memory. In other words, droughts were likely to be followed by other, and in some cases more severe, droughts. By solving this puzzle, Hurst was able to provide a more precise
answer for the dam builders, that ended up taming the Nile. As Hurst’s theory gained acceptance, it proved to be widely applicable to a slew of complex natural phenomena such as width of tree rings, the number of sun spots, annual pattern of rainfall, as well as certain man made phenomena such as the price behavior of commodities, stock price fluctuations, international crude oil prices and economic indicators. We intend to apply this theory and the extensions introduced by Vogel & McMahon (Vogel & McMahon 1996), to retirement planning.

The parallels between dam construction and wealth building for retirement are numerous. Dams are designed to store water during abundant years and to even out the outflows from a river system. The hydrologists’ challenge is to determine the exact size of the dam from historic flow patterns of the river. If the dam is too low then it ends up having very little buffering effect. If on the other hand it is too high, then it ends up being too expensive to construct and inefficient to run. Ideally, the dam needs to be just high enough that it serves as a regulator, minimizing or eliminating flood damage during wet years, and mitigating the effects of drought during the dry years. Similarly, retirement savings are intended to build wealth so as to generate an even and long lasting stream of income. Just like the Nile’s fickleness, the financial markets present unpredictable year over year variability which Hurst’s formula proved adept at explaining (Mandelbrot 1963). We will exploit these and other similarities between the two systems to adapt the mathematics of dam building to retirement planning.
Impact of volatility and consumption on wealth

The first savings related insight from the mathematics of dam construction comes from a non-dimensional index introduced by Hurst (1951), that is reflective of system resiliency:

\[ m = \frac{(1 - \alpha)\mu}{\sigma} \]  

\( \alpha \): annual consumption as a fraction of mean annual income generated  
\( \mu \): mean annual income generated by the portfolio  
\( \sigma \): standard deviation of annual incomes

This formula calibrates the amount of income that is not consumed by the variation in the income stream generated by the portfolio. In plain English, the higher the percentage of income withdrawn, and/or the higher the standard deviation in this income stream, the less resilient the savings plan. In other words, as the value of \( m \) gets closer to zero, the system becomes progressively more likely to fail (see Chart I & II).

There are two lessons to be learned from this deceptively simple formula. First, volatility of the income stream has a significant impact on wealth generation and wealth preservation. Reducing the volatility of a given income stream can enhance wealth. Alternatively, investors seeking higher returns from their investments should take risk in proportionate levels so as to prevent a decline in the resiliency of the savings system. We can best demonstrate the validity of this point with an example. Two return streams with an identical arithmetic return of 7.5%, and standard deviations of 10% and 15%, would
generate an average compound return of 7% and 6.4% respectively (Fiduciary Insight 2006). This example clearly demonstrates the impact of variation on compound return and terminal wealth. Therefore, it is safe to conclude that portfolio volatility is one of the critical levers in retirement planning.

The second lesson has to do with the amount withdrawn from a savings plan. Conventional wisdom within the financial industry relates withdrawals from a retirement system to the total amount saved. However, studies of river systems and dams have demonstrated that the size of the reservoir should not be used as a guide in planning the outflows. This unsound strategy is more likely to result in failed dams and dry river beds. Instead, the outflows should be guided by the average amount of inflows. Analogously, in retirement planning, the higher the withdrawal percentage relative to income generated, the less resilient the savings system. Therefore, to reduce the likelihood of failure, withdrawals from the system must be kept at a level that is proportionate, and preferably low, relative to the average income generated by the system.

**How much savings is enough?**

Although system resiliency is important, it is dependent on the amount of savings. So, our next challenge is to address this key question: how much does one need to save to generate a given level of income during retirement, with a certain level of confidence?

Through the study of river systems, hydrologists have developed mathematical models that interlink storage capacity, reliability and yield (SRY) of dams. The mathematics of
saving which we introduce here will once again borrow extensively from the science of hydrology. A caveat is that the models developed for dam construction rely on independent inflows, as dams are replenished by rainfall and melting snow. In contrast, the health of retirement savings is entirely dependent on the amount of remaining savings. To meet this requirement of independence, we need to separate savings from the income stream. We will accomplish this by making one simple assumption that the annual inflows are generated by a separate savings plan. As long as we separate the inflows from any subsequent savings that are necessary to dampen volatility, we can accurately utilize the SRY models. Combining the dollar amount of the savings required to support the inflows, with the savings required to dampen market volatility, provides the total savings one would need to assure a consistent income stream at retirement.

First we will address the amount of savings that are required to dampen volatility ($S_{vol}$). Over the retirement lifetime ($n$) of an individual, the total inflows can be expressed as the sum of annual income flows ($\mu$) plus a random component due to market related uncertainty.

$$\sum x(n) = n\mu + t_{q}\sigma n^{1/2}$$

(II)

$n$: life span in years

$\mu$: average annual income flows

$t_{q}\sigma n^{1/2}$: q percentile of a standardized normal random distribution
Since total yield \( y \) realized by the individual will be dependent on the inflows, we can express the total yield as:

\[
y = \alpha n \mu \quad \text{(III)}
\]

Therefore, total savings \( S_{\text{vol}}(n) \) can be expressed as the difference between total yield \( y \) and total inflows \( \sum x(n) \).

\[
S_{\text{vol}}(n) = \alpha n \mu - \sum x(n) \quad \text{(IV)}
\]

As the reader can surmise, the magnitude of savings will be determined by the longest sequence of drawdowns. In this context, drawdowns occur when the income generated by the savings \( \sum x(n) \) is less than the outflows needed by the individual \( \alpha n \mu \). At this point, iterative techniques can be used to calculate the answer to equation IV. However, these techniques are time consuming and path dependent. A faster and mathematically more elegant answer to the critical drawdown sequence \( n^* \) can be obtained by differentiating equation (IV) with respect to \( n \), and equating it to 0 (Gould 1964).

\[
n^* = \left[ \frac{t_\sigma \sigma}{2(1-\alpha)\mu} \right]^2 \quad \text{(V)}
\]

If we substitute Hurst’s resilience formula (I) into the above equation (V) we end up with:
\[ n^* = \frac{t_q^2}{4m^2} \]  \hspace{1cm} (VI)

With some additional algebra, which substitutes equation VI in place of \( n \) in equation IV, we derive the optimal savings amount to be:

\[ S_{\text{vol}} = \frac{t_q^2 \alpha}{4m} \]  \hspace{1cm} (VII)

Alternatively, for a more intuitive version, we can re-write this equation as follows:

\[ S_{\text{vol}} = n^*(1 - \alpha)\mu \]  \hspace{1cm} (VIII)

The above equation (VIII) states that the size of savings is equivalent to the critical drawdown sequence (\( n^* \)) multiplied by the average amount of annual income plowed back into the savings pool. In effect, \( n^* \) serves as a multiplier for determining how much needs to be saved to buffer volatility.

Up to this point, we have introduced mathematical formulas that assume a certain inflow and calculate the savings necessary to prevent drawdowns caused by market volatility. However, there is one more remaining step. We have to add the savings necessary to support the inflows (\( S_i \)). Since market fluctuations are taken into account by \( S_{\text{vol}} \), the math for \( S_i \) is straightforward:
\[ S_i = \mu \times \left( \frac{1 - \frac{1}{(1 + r)^n}}{r} \right) \]  \hspace{1cm} (IX)

\( \mu = \) income stream generated by savings \( S_i \)

\( r = \) estimated average market return

\( n = \) years of life span during retirement

Therefore, total estimated savings (\( S_{tot} \)) to support a certain income stream is:

\[ S_{tot} = S_i + S_{val} \]  \hspace{1cm} (X)

The savings amount provided by these formulas assumes that the individual would not touch the principal savings pool, other than using it as a buffer during drawdown years, and would pass it on to his/her heirs. If such is not the case, the equations can be adjusted accordingly.

The most critical insight that can be derived from the above math is that the health of the savings plan is entirely dependent on the sequence and timing of market drawdowns. Conventional wisdom, which is based on planning savings on average returns, is likely to leave the entity short of income during the worst possible time. Instead, savings should be planned on the longest sequence of drawdowns.
The level of confidence required is also a critical lever in planning for savings. The more assurance we require that we’d never run out of savings, the larger the multiplier and hence the savings pool $S_{vol}$.

**Some real-world answers**

An example might help to further clarify some of the concepts introduced. For instance, an individual who requires $80,000 to support his lifestyle, at the 80% withdrawal level, would need $100,000 in income. Assuming the standard deviation of this income stream is $120,000, we’d obtain a resilience index of:

$$m = \frac{(1 - 0.8) \times 100,000}{120,000} = 0.1667$$

If the income flows generated by the market are normally distributed, and we require 95% reliability that the $S_{vol}$ savings pool will not fail, then the critical drawdown multiplier is:

$$n^* = \frac{1.65^2}{4 \times 0.1667^2} = 24.69$$

This results in a savings pool $S_{vol}$ necessary to offset volatility:

$$S_{vol} = 24.69 \times (1 - 0.8) \times 100,000 = 493,766$$
The total savings required by this individual is $1,475,581, assuming a 20 year time frame and 8% return on Si.

If returns could be guaranteed, this individual would be able to meet his income needs with $981,815 in savings. However, the impact of volatility on returns expands the savings by 33%. This example once again demonstrates the impact of volatility on wealth and retirement income.

**Conclusion**

Historically, savings and retirement planning have relied on techniques that proved to be failure prone at best, and completely inadequate under the worst circumstances. Granted, some of these adverse consequences came about due to the poor applications of the current methods; but more importantly, the methods themselves proved to be deficient.

In this paper, we adapted an approach for savings and retirement planning that originated in hydrology. On the surface, the mathematics of dam building appears to bear little relevance to finance. However, a more in depth examination reveals that the mathematical relationships that are reflective of the behavior of rivers are also effective in modeling other natural and man-made phenomena.

Our goal was twofold: to develop a simple and understandable technique, and to draw lessons from this framework. We strived for simplicity in order to enhance insight. In
the process, we demonstrated that certain assumptions, such as basing withdrawals on principal, are flawed.

Much work remains to develop this new path of exploration, but we hope that the dialogue to improve the state of the science in savings and retirement planning will benefit from this new perspective.
Tables Exhibits and Charts

- Chart I. -

Resilience of Savings Plans

- Chart II -

Impact of Income Volatility on Resilience
Footnotes:

(1) According to PBGC statistics from 1986 to 2004, 101,000 defined benefit pension plans were terminated.
(2) Securities Industry Association Retirement Study, June 2006
(3) See Vogel & McMahon pg 80
(4) Creating a separate savings plan for the inflows assures that the inflows are not dependent on subsequent savings. In other words, we are assuming that two savings plans exist. The first one is designed to support the inflows, and the second one is designed to buffer the volatility of the inflows. Then to come up with a total savings amount we combine the dollar value of these two plans.
(5) For normally distributed inflows $t_q = z_q$, which can be calculated as:
$$Z_q = 4.91 * (q^{0.141} - (1-q)^{0.141})$$
References

Richard Vogel, T. A. McMahon, , February 1996 “Approximate reliability and resilience indices of over-year reservoirs fed by AR(1) Gamma and normal flows”, Hydrological Sciences Journal
Fiduciary Insight, July 2006, “How risk management adds wealth”